Week 9 Worksheet
Even More Perturbation Theory

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Exercise 1. Stark Effect in Hydrogen. When an atom is placed in a uniform electric field $\mathbf{E}_{\text{ext}}$, the energy levels are shifted. This is known as the **Stark effect**. You'll analyze the Stark effect for the $n = 1$ and $n = 2$ states of hydrogen. Suppose $\mathbf{E}_{\text{ext}} = E_{\text{ext}}\hat{z}$, so that

$$H' = eE_{\text{ext}} r \cos\theta$$

is the perturbation of the hamiltonian for the electron, where $H^0 = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}$.

a) Show that the ground state energy is unchanged at first order.

b) How much degeneracy does the first excited state have? List the degenerate states.

c) Determine the first-order corrections to the energy. Into how many levels does E_2 split?

   Hint: All W_{ij} are 0 except for two, and you can avoid doing all of the zero integrals in this problem by using symmetry and selection rules. You'll need the following

$$\begin{aligned}\psi_{210} &= \frac{1}{2\sqrt{6}} a^{-3/2} \frac{r}{a} e^{-r/2a} \sqrt{\frac{3}{4\pi}} \cos\theta \\ \psi_{200} &= \frac{1}{\sqrt{2}} a^{-3/2} \left(1 - \frac{r}{2a}\right) e^{-r/2a} \frac{1}{2\sqrt{\pi}}.\end{aligned}$$

d) What are the "good" wavefunctions for (b)? Find the expectation value of the electric dipole moment in each of these states.

Exercise 2. A free particle of mass m is constrained to move on a ring of circumference L, so that $\psi(x) = \psi(x + L)$. We then add a perturbation

$$H' = V_0 \cos(2\pi x/L).$$

a) Show that the unperturbed eigenstates can be written

$$\psi_n^0(x) = \frac{1}{\sqrt{L}} e^{2\pi i n x/L},$$

   where $n \in \mathbb{Z}$. Show further that all of the states are two-fold degenerate for $n \neq 0$.

b) Find a general expression for the matrix elements

$$H'_{mn} = \langle \psi_m^0 | H' | \psi_n^0 \rangle.$$

c) Consider the degenerate pair of states with $n \in \{-1, 1\}$. Show that there are no first order corrections; hence, the degeneracy does not lift at first order.

d) We have to use second order degenerate perturbation theory to tackle this problem. Show, by using the equation $H^0\psi^1 + H'\psi^0 = E^0\psi^1 + E^1\psi^0$, where $\psi^0 = \alpha\psi_a^0 + \beta\psi_b^0$, that

$$\psi^1 = \sum_{m \notin \{a,b\}} \frac{\alpha H'_{ma} + \beta H'_{mb}}{E^0 - E_m^0} \psi_m^0$$

for a two-fold degenerate system.

e) It turns out that α and β are given by the components of the eigenvectors of the matrix $W^{(2)}$, where

$$(W^{(2)})_{ij} = \sum_{m \notin \{a,b\}} \frac{H'_{im} H'_{mj}}{E^0 - E_m^0}.$$

The second order corrections to the energies are the eigenvalues of this matrix. Construct the matrix $W^{(2)}$, and show that the degeneracy lifts at second order for the states $n \in \{-1, 1\}$. What are the good linear combinations of the states for $n \in \{-1, 1\}$?

f) What are the energies at second order for these states?

g) By considering the terms of order λ^2 in degenerate perturbation theory for two-fold degeneracy, prove the (unproven) assertions of part (e).